Reproductive strong solutions of Navier-Stokes equations with non homogeneous boundary conditions

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Abstract

The object of the present paper is to show the existence and the uniqueness of a reproductive strong solution of the Navier-Stokes equations, i.e. the solution $\boldsymbol{u}$ belongs to $\mathbf{L}^{\infty}(0,T;V)\cap\mathbf{L}^{2}\left(0,T;\mathbf{H}^{2}(\Omega)\right)$ and satisfies the property $\boldsymbol{u}(\boldsymbol{x},T)=\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})$. One considers the case of an incompressible fluid in two dimensions with nonhomogeneous boundary conditions, and external forces are neglected.

Key Words: Navier-Stokes equations, incompressible fluid, reproductive solution, nonhomogeneous boundary conditions.

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1 Introduction and notations

Let Ω be an open and bounded domain of $\mathbb{R}^2$, with a sufficiently smooth boundary Γ; and let us consider the Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} - \nu\triangle\boldsymbol{v} + \boldsymbol{v}.\nabla\boldsymbol{v} + \nabla p = 0 & \text{in} \quad Q_T=\Omega\times\,]0,T[\,, \\ \text{div }\boldsymbol{v}=0 & \text{in} \quad Q_T, \\ \boldsymbol{v}=\boldsymbol{g} & \text{on} \quad \Sigma_T=\Gamma\times\,]0,T[\,, \\ \boldsymbol{v}(0)=\boldsymbol{v}_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

where $\boldsymbol{g}$, $\boldsymbol{v}_0$ and $T>0$ are given.We suppose that :

$$\text{div }\boldsymbol{v}_0=0 \ \text{ in } \ \Omega, \quad \boldsymbol{v}_0.\boldsymbol{n}=0 \quad \text{on} \quad \Gamma, \tag{2}$$

and

$$\boldsymbol{g}.\boldsymbol{n}=0 \quad \text{on} \quad \Sigma_T. \tag{3}$$

One is interested on one hand by the existence of strong solutions of system (1). On the other hand, one seeks data conditions to establish the existence of a reproductive solution generalizing the concept of a periodic solution. Kaniel and

Shinbrot [5] showed the existence of these solutions for system (1) in dimensions 2 and 3 with external forces but zero boundary condition i.e. $\boldsymbol{g} = 0$. With another approach using semigroups, one can also point out the work of Takeshita [10] in dimension 2.

We need to introduce the following functional spaces, with r and s positive numbers:

$$\mathbf{H}^{r,s}(Q_T) \ = \mathbf{L}^2\left(]0,T[\,;\mathbf{H}^r(\Omega)\right) \cap \mathbf{H}^s\left(]0,T[\,;\mathbf{L}^2(\Omega)\right)$$

These are Hilbert spaces for the norm

$$\|\boldsymbol{v}\|_{\mathbf{H}^{r,s}(Q_T)} \ = \left(\int_0^T \|\boldsymbol{v}(t)\|^2_{\mathbf{H}^r(\Omega)}\,dt + \|\boldsymbol{v}\|^2_{\mathbf{H}^s(]0,T[;\mathbf{L}^2(\Omega))}\right)^{1/2}.$$

Let us recall that for $s = 1$, for example,

$$\|\boldsymbol{v}\|_{\mathbf{H}^1(]0,T[;\mathbf{L}^2(\Omega))} \ = \left[\int_0^T \left(\|\boldsymbol{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\mathbf{L}^2(\Omega)}\right) dt\right]^{1/2}.$$

In the same manner one defines spaces $\mathbf{H}^{r,s}(\Sigma_T)$.
We now introduce the following spaces:

$$\begin{aligned}
\mathcal{V} &= \left\{\boldsymbol{v} \in \mathcal{D}(\Omega)^2;\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \quad \Omega\right\},\\
\mathrm{H} &= \left\{\boldsymbol{v} \in \mathbf{L}^2(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \quad \Omega,\ \boldsymbol{v}.\boldsymbol{n} = 0 \text{ on } \quad \Gamma\right\},\\
V &= \left\{\boldsymbol{v} \in \mathbf{H}^1_0(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \quad \Omega\right\},
\end{aligned}$$

Let us recall that $\mathcal{V}$ is dense in H and V for their respective topologies.

Here, $\mathcal{D}(\Omega)$ is the class of $\mathcal{C}^\infty$ functions with compact support in Ω. The notations $(.,.)$ et $((.,.))$ indicate the scalar products in $\mathbf{L}^2(\Omega)$ and in $\mathbf{H}^1_0(\Omega)$ respectively, and $|.|$ et $\|.\|$ the associated norms.

In the order to solve problem (1), we will have to remove boundary condition $\boldsymbol{g}$. and consider a new problem with zero boundary condition. We note that if $\boldsymbol{v} \in \mathbf{H}^{2,1}(Q_T)$ is solution of (1), then thanks to the Aubin compactness lemma (see J.L. Lions [8], R. Temam [11]) one will have

$$\boldsymbol{v} \in \mathcal{C}^0\left([0,T]\,;\mathbf{H}^1(\Omega)\right) \hookrightarrow \mathcal{C}^0\left([0,T]\,;\mathbf{H}^{1/2}(\Gamma)\right)$$

So that a necessary condition for $\boldsymbol{v}$ to exist is that:

$$\boldsymbol{g}\left(\boldsymbol{x},0\right) = \boldsymbol{v}_0\left(\boldsymbol{x}\right), \quad \boldsymbol{x} \in \Gamma. \tag{4}$$

Combining (2)-(4), one has:

$$\boldsymbol{g}.\boldsymbol{n} = 0 \quad \text{on} \;\; \Gamma \times [0, T[\,.$$

The following lemma allows us to state hypotheses on $\boldsymbol{g}$ (voir Lions-Magenes [7]).

Lemma 1.1. *Suppose that (4) takes place and let*

$$\boldsymbol{g} \in \mathbf{H}^{3/2,3/4}(\Sigma_T), \qquad \boldsymbol{v}_0 \in \mathbf{H}^1(\Omega). \tag{5}$$

Then there exists a function $\mathbf{R} \in \mathbf{H}^{2,1}(Q_T)$ *such that*

$$\mathbf{R} = \boldsymbol{g} \;\; \text{on} \quad \Sigma_T \;\text{et}\;\; \mathbf{R}\,(0) \; = \boldsymbol{v}_0 \;\text{in} \quad \Omega, \tag{6}$$

and satisfying the estimates

$$\|\mathbf{R}\|_{\mathbf{H}^{2,1}(Q_T)} \leq C\left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}\right).\Box \tag{7}$$

We now consider the problem:

For a given $\boldsymbol{g}$ verifying (5), one seeks ($\boldsymbol{u}$, q) which satisfies

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu \triangle \boldsymbol{u} + \nabla q = 0 & \text{in} \qquad Q_T, \\ \text{div } \boldsymbol{u} = \text{div } \mathbf{R} & \text{in} \qquad Q_T, \\ \boldsymbol{u} = 0 & \text{on} \qquad \Sigma_T, \\ \boldsymbol{u}(0) = \mathbf{0} & \text{in} \qquad \Omega. \end{cases} \tag{8}$$

The following proposition holds (see Dautray-Lions [2] , O. A. Ladyzhenskaya [6] , V.A. Solonnikov [9]) :

Proposition 1.2. *We suppose that (5)holds,*

$$\text{div } \boldsymbol{v}_0 = 0 \;\; \textit{on} \;\; \Omega, \;\; \boldsymbol{v}_0.\boldsymbol{n} = 0 \;\; \textit{in} \;\; \Gamma, \textit{and} \; \boldsymbol{g}.\boldsymbol{n} = 0 \;\; \textit{in} \;\; \Sigma_T. \tag{9}$$

Then problem (8) has an unique solution ($\boldsymbol{u}$, q) *such that*

$$\boldsymbol{u} \in \mathbf{H}^{2,1}(Q_T), \qquad\qquad q \in L^2\left(0, T; H^1(\Omega)^2\right)$$

with the estimates

$$\|\boldsymbol{u}\|_{\mathbf{H}^{2,1}(Q_T)} + \|q\|_{L^2(0,T;H^1(\Omega)^2)} \;\leq C\left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}\right).\Box \tag{10}$$

Thus the function defined by

$$\mathbf{G} = \mathbf{R} - \ \boldsymbol{u} \qquad \text{in } Q_T \tag{11}$$

satisfies the estimates (7) and

$$\text{div } \mathbf{G} = 0 \qquad \text{in } Q_T, \tag{12}$$

$$\mathbf{G} = \boldsymbol{g} \qquad \text{on } \Sigma_T, \tag{13}$$

$$\mathbf{G}\left(\boldsymbol{x},0\right) \ = \boldsymbol{v}\left(\boldsymbol{x},0\right) \qquad \boldsymbol{x} \ \in \Omega. \tag{14}$$

This yields the following lemma:

Lemma 1.3. *Let* $\boldsymbol{g}$ and $\boldsymbol{v}_0$ *satisfy* (4), (5) and (9). *Then there exists* $\mathbf{G} \in \mathbf{H}^{2,1}(Q_T)$ *satisfying (12)-(14) and the estimate*

$$\|\mathbf{G}\|_{\mathbf{H}^{2,1}(Q_T)} \leq C\left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}\ \right).\square$$

Moreover, one has the next lemma

Lemma 1.4. *Let* $\varepsilon > 0$,*and let* $\boldsymbol{g}$ and $\boldsymbol{v}_0$ *satisfy the hypotheses of lemma 1.3. Then there exists* $\mathbf{G}_\varepsilon \in \mathbf{H}^{2,1}(Q_T)$ *such that*

$$\text{div } \mathbf{G}_\varepsilon = 0 \quad \text{in} \quad Q_T,$$

$$\mathbf{G}_\varepsilon = \boldsymbol{g} \qquad \text{on} \qquad \Sigma_T,$$

$$\|\mathbf{G}_\varepsilon\left(.,0\right)\|_{\mathbf{H}^1(\Omega)} \leq C_\varepsilon \|\mathbf{G}\left(.,0\right)\|_{\mathbf{H}^1(\Omega)}$$

and

$$\forall \boldsymbol{v} \in V, \quad \left|b\left(\boldsymbol{v}, \mathbf{G}_\varepsilon\left(t\right), \boldsymbol{v}\right)\right| \leq \beta(\varepsilon, t)\,\|\nabla \boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}$$

with

$$\sup_{t\in[0,T]} \beta(\varepsilon, t) \to 0 \ \textit{when} \ \ \varepsilon \to 0.$$

Moreover, there exists an increasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$, *not depending on* ε, *such that*

$$\|\mathbf{G}_\varepsilon\|_{\mathbf{H}^{2,1}(Q_T)} \leq L\left(\frac{\varepsilon}{\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}}\right)\left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}\right).$$

Proof.

i) Step 1 : One takes up again the Hopf construction (see Girault & Raviart [4], Temam [11], Lions [8], Galdi [3]).

ii) Step 2 : The open domain Ω being smooth, and since div $\mathbf{G}_\varepsilon = 0$ in Q_T and

$\mathbf{G}.\boldsymbol{n} = 0$ on $\Gamma \times [0,T[$, there exists, for all $t \in [0,T[$, a function ψ depending on $\boldsymbol{x}$ and t, such that

$$\mathbf{G} = \mathbf{rot}\ \psi \quad \text{in} \quad \Omega \times [0,T\ ]$$

with $\psi = 0$ on $\Gamma \times [0,T[$, $\psi \in \mathbf{L}^2\left(0,T;\mathbf{H}^3(\Omega)\right)$, $\dfrac{\partial \psi}{\partial t} \in \mathbf{L}^2\left(0,T;\mathbf{H}^1(\Omega)\right)$ and satisfying the estimate

$$\|\psi\|_{\mathbf{L}^2(0,T;\mathbf{H}^3(\Omega))} + \|\psi_t\|_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))} \le C\,\|\mathbf{G}\|_{\mathbf{H}^{2,1}(Q_T)}\,. \tag{15}$$

iii) Step 3 : Let

$$\mathbf{G}^{\varepsilon} = \mathbf{rot}\,(\theta_{\varepsilon}\ \psi)\,.$$

One deduces from the properties of θ_{ε}, for $j = 1,2$:

$$\left|\mathbf{G}_j^{\varepsilon}(x,t)\right| \le C\left(\frac{\varepsilon}{\rho\,(x)}\,|\psi(x,t)| + |\nabla\psi(x,t)|\right) \qquad \text{if} \qquad \rho(x) \le 2\delta(\varepsilon)$$

and $\mathbf{G}_j^{\varepsilon} = 0$ if $\rho(x) > 2\delta(\varepsilon)$.
We note that

$$\psi \in C\left([0,T]\,;\mathbf{H}^2(\Omega)\right) \hookrightarrow C\left([0,T]\,;\mathbf{L}^{\infty}(\Omega)\right).$$

Therefore,

$$\left|\mathbf{G}_j^{\varepsilon}(x,t)\right| \le C\left(\frac{\varepsilon}{\rho\,(x)} + |\nabla\psi(x,t)|\right) \qquad \text{if} \qquad \rho(x) \le 2\delta(\varepsilon).$$

Thus, for all $\boldsymbol{v} \in \mathbf{H}_0^1(\Omega)$,

$$\left\|\boldsymbol{v}_i \mathbf{G}_j^{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)} \le C\left[\varepsilon\left\|\frac{\boldsymbol{v}_i}{\rho}\right\|_{\mathbf{L}^2(\Omega)} + \left(\int\limits_{\rho(x)\le 2\delta(\varepsilon)} \boldsymbol{v}_i^2.\,|\nabla\psi|^2\,dx\right)^{1/2}\right]$$

$$\left\|\boldsymbol{v}_i \mathbf{G}_j^{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)} \le C\varepsilon\,\|\nabla\boldsymbol{v}_i\|_{\mathbf{L}^2(\Omega)} + C\,\|\nabla\boldsymbol{v}_i\|_{\mathbf{L}^2(\Omega)} \times \left(\int\limits_{\rho(x)\le 2\delta(\varepsilon)} |\nabla\psi|^3\,dx\right)^{1/3}$$

Setting

$$\beta(\varepsilon,t) = \left(\int\limits_{\rho(x)\le 2\delta(\varepsilon)} |\nabla\psi|^3\,dx\right)^{1/3},$$

it's clear that

$$\lim_{\varepsilon \to 0} \beta(\varepsilon, t) = 0 \ uniformly\ on\ \ [0, T] \, .$$

The second inequality of lemma 1.4 is a consequence of Hölder inequality. The first inequality follows from Hardy inequality for $\mathbf{H}_0^1(\Omega)$-functions and properties of θ_ε .□

2 Existence of strong solutions

Let us make a change of the unknown function in problem (1), by setting

$$\boldsymbol{u} = \boldsymbol{v} - \mathbf{G}_\varepsilon, \qquad\qquad \boldsymbol{u}_0 = \boldsymbol{v}_0 - \mathbf{G}_\varepsilon\,(.,0)\,,$$

where $\mathbf{G}_\varepsilon$ is the function given by lemma 1.4. Problem (1) then becomes:

$$\left\{ \begin{array}{lll} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu \triangle \boldsymbol{u} + \boldsymbol{u}.\nabla \boldsymbol{u} + \boldsymbol{u}.\nabla \mathbf{G}_\varepsilon + \mathbf{G}_\varepsilon.\nabla \boldsymbol{u} + \nabla p = \boldsymbol{f}_\varepsilon & \text{in} & Q_T \\ \text{div}\ \boldsymbol{u} = 0 & \text{in} & Q_T \\ \boldsymbol{u} = 0 & \text{on} & \Sigma_T \\ \boldsymbol{u}(0) = \boldsymbol{u}_0^\varepsilon & \text{in} & \Omega \end{array} \right. \tag{16}$$

with

$$\boldsymbol{f}_\varepsilon = -\frac{\partial \mathbf{G}_\varepsilon}{\partial t} + \nu \triangle \mathbf{G}_\varepsilon - \mathbf{G}_\varepsilon.\nabla \mathbf{G}_\varepsilon \quad \text{and} \quad \boldsymbol{u}_0^\varepsilon = \boldsymbol{v}_0 - \mathbf{G}_\varepsilon\,(.,0)\,. \tag{17}$$

We note that $\boldsymbol{u}_0^\varepsilon \in V$ and

$$\|\boldsymbol{u}_0^\varepsilon\|_{\mathbf{H}^1(\Omega)} \leq C_\varepsilon \left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \tag{18}$$

Moreover, $\boldsymbol{f}_\varepsilon \in \mathbf{L}^2\left(0, T; \mathbf{L}^2(\Omega)\right)$ and

$$\|\boldsymbol{f}_\varepsilon\|_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} \leq C_\varepsilon \left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \tag{19}$$

Now we are able to announce and to establish the following theorem :

Theorem 2.1. *Let* $\boldsymbol{v}_0$ and $\boldsymbol{g}$ *satisfy the hypotheses of lemma 1.3. Then problem (16) has a unique solution* $(\boldsymbol{u}, p)$ *such that*

$$\boldsymbol{u} \in \mathbf{L}^2\left(0, T; \mathbf{H}^2(\Omega)\right) \cap \mathbf{L}^\infty\left(0, T; V\right), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in \mathbf{L}^2\left(0, T; \mathbf{H}\right), \quad p \in \mathbf{L}^2\left(0, T; H^1(\Omega)\right),$$

p being unique up to an $\mathbf{L}^2(0, T)$*-function of the single variable t.*

Proof.

2.1 *Approximate solutions*

We use the Galerkin method. Let $m \in \mathbb{N}^*$ and $\boldsymbol{u}_{0m} \in \langle \boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_m \rangle$ such that

$$\boldsymbol{u}_{0m} \to \boldsymbol{u}_0^\varepsilon \text{ in } V, \text{ if } m \to \infty,$$

where $\boldsymbol{w}_j$ are the Stokes operator eigenfunctions . For each m, one defines an approximate solution of (16) by :

$$\left\{ \begin{array}{c} \boldsymbol{u}_m(t) = \sum\limits_{j=1}^{m} g_{jm}(t) \boldsymbol{w}_j \\ \left(\boldsymbol{u}'_m(t), \boldsymbol{w}_j\right) + \nu \left(\left(\boldsymbol{u}_m(t), \boldsymbol{w}_j\right)\right) + b\left(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j\right) \\ +b\left(\boldsymbol{u}_m(t), \mathbf{G}_\varepsilon(t), \boldsymbol{w}_j\right) + b\left(\mathbf{G}_\varepsilon(t), \boldsymbol{u}_m(t), \boldsymbol{w}_j\right) = \left(\boldsymbol{f}_\varepsilon(t), \boldsymbol{w}_j\right) \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \ \ j = 1, ..., m \end{array} \right. \tag{20}$$

This is a nonlinear differential system of m equations in m unknowns g_{jm}, $j = 1, ..., m$:

$\sum_{i=1}^{m} \left(\boldsymbol{w}_i, \boldsymbol{w}_j\right) g'_{im}(t) + \nu \sum_{i=1}^{m} \left(\left(\boldsymbol{w}_i, \boldsymbol{w}_j\right)\right) g_{im}(t) + \sum_{i,l=1}^{m} b\left(\boldsymbol{w}_i, \boldsymbol{w}_l, \boldsymbol{w}_j\right) g_{im}(t) g_{lm}(t) +$
$+ \sum_{i=1}^{m} \left[b\left(\boldsymbol{w}_i, \mathbf{G}_\varepsilon(t), \boldsymbol{w}_j\right) g_{im}(t) + b\left(\mathbf{G}_\varepsilon(t), \boldsymbol{w}_i, \boldsymbol{w}_j\right) g_{im}(t)\right] = \left(\boldsymbol{f}_\varepsilon(t), \boldsymbol{w}_j\right),$
$j = 1, ..., m$

2.2 *Estimates I*

Let us multiply (20) by $g_{jm}(t)$ and sum over j :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\boldsymbol{u}_m(t)\right|^2 + \nu \left\|\boldsymbol{u}_m(t)\right\|^2 &= -b\left(\boldsymbol{u}_m(t), \mathbf{G}_\varepsilon(t), \boldsymbol{u}_m(t)\right) + \left(\boldsymbol{f}_\varepsilon(t), \boldsymbol{u}_m(t)\right) \\ &\leq \left|\boldsymbol{f}_\varepsilon(t)\right| \left\|\boldsymbol{u}_m(t)\right\| + \left|b\left(\boldsymbol{u}_m(t), \mathbf{G}_\varepsilon(t), \boldsymbol{u}_m(t)\right)\right| \end{aligned}$$

One deduces from lemma 1.4 that :

$$\frac{1}{2}\frac{d}{dt}\left|\boldsymbol{u}_m(t)\right|^2 + \frac{\nu}{2}\left\|\boldsymbol{u}_m(t)\right\|^2 \leq \frac{1}{2\nu C^2(\Omega)}\left|\boldsymbol{f}_\varepsilon(t)\right|^2 + \beta(\varepsilon, t)\left\|\boldsymbol{u}_m(t)\right\|^2.$$

As $\sup\limits_{t \in [0,T]} \beta(\varepsilon, t) \to 0$ when $\varepsilon \to 0$, for a fixed and small $\varepsilon > 0$, one has:

$$\frac{d}{dt}\left|\boldsymbol{u}_m(t)\right|^2 + \frac{\nu}{2}\left\|\boldsymbol{u}_m(t)\right\|^2 \leq \frac{1}{\nu C^2(\Omega)}\left|\boldsymbol{f}_\varepsilon(t)\right|^2. \tag{21}$$

Integrating (21) from 0 to s, one deduces that:

$$\begin{aligned} \left|\boldsymbol{u}_m(s)\right|^2 &\leq \left|\boldsymbol{u}_{0m}\right|^2 + \frac{1}{\nu C^2(\Omega)} \int_0^s \left|\boldsymbol{f}_\varepsilon(t)\right|^2 dt \\ &\leq \left|\boldsymbol{u}_0^\varepsilon\right|^2 + \frac{1}{\nu C^2(\Omega)} \left\|\boldsymbol{f}_\varepsilon(t)\right\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} \\ &\leq C_\varepsilon \left(\left\|\boldsymbol{g}\right\|^2_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \left\|\boldsymbol{v}_0\right\|^2_{\mathbf{H}^1(\Omega)}\right) \end{aligned}$$

according to (18) and (20). Therefore

$$\boldsymbol{u}_m \quad \in \ \mathbf{L}^{\infty}(0,T;\mathbf{H}), \tag{22}$$

and $\{\boldsymbol{u}_m\}$ is an equibounded sequence in $\mathbf{L}^{\infty}(0,T;\mathbf{H})$.

Next, thanks to (21), one has:

$$\boldsymbol{u}_m \quad \in \ \mathbf{L}^{2}(0,T;V), \tag{23}$$

and the sequence $\{\boldsymbol{u}_m\}$ is equibounded in $\mathbf{L}^2(0,T;\mathbf{V})$.

2.3 *Estimates II*

Let us multiply (20) by $\lambda_j g_{jm}(t)$ and sum over j :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{u}_m(t)\right\|^2+\nu\left|A\boldsymbol{u}_m(t)\right|^2+b\left(\boldsymbol{u}_m(t),\boldsymbol{u}_m(t),\,A\boldsymbol{u}_m(t)\right)+\\&b\left(\mathbf{G}_\varepsilon(t),\boldsymbol{u}_m(t),\,A\boldsymbol{u}_m(t)\right)+b\left(\boldsymbol{u}_m(t),\mathbf{G}_\varepsilon(t),\,A\boldsymbol{u}_m(t)\right)=\left(\boldsymbol{f}_\varepsilon,\,A\boldsymbol{u}_m(t)\right)\end{aligned} \tag{24}$$

where A is the Stokes operator. Let us begin by considering the nonlinear terms. For the first term, thanks to the Gagliardo-Nirenberg inequality one has

$$\begin{aligned}\left|b\left(\boldsymbol{u}_m(t),\boldsymbol{u}_m(t),\ A\boldsymbol{u}_m(t)\right)\right| &\leq \left\|\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left|A\boldsymbol{u}_m(t)\right|\\&\leq C\left|\boldsymbol{u}_m(t)\right|^{1/2}\left\|\boldsymbol{u}_m(t)\right\|\left|A\boldsymbol{u}_m(t)\right|^{3/2}\\&\leq C\left\|\boldsymbol{u}_m(t)\right\|^4+\frac{\nu}{8}\left|A\boldsymbol{u}_m(t)\right|^2.\end{aligned}$$

In the same way,

$$\begin{aligned}\left|b\left(\mathbf{G}_\varepsilon(t),\boldsymbol{u}_m(t),\,A\boldsymbol{u}_m(t)\right)\right| &\leq \left\|\mathbf{G}_\varepsilon(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left|A\boldsymbol{u}_m(t)\right|\\&\leq C\left\|\mathbf{G}_\varepsilon(t)\right\|_{\mathbf{H}^1(\Omega)}\left\|\boldsymbol{u}_m(t)\right\|^{1/2}\left|A\boldsymbol{u}_m(t)\right|^{3/2}\\&\leq C\left\|\mathbf{G}_\varepsilon(t)\right\|^4_{\mathbf{H}^1(\Omega)}\left\|\boldsymbol{u}_m(t)\right\|^2+\frac{\nu}{8}\left|A\boldsymbol{u}_m(t)\right|^2.\end{aligned}$$

We remark that, according to lemma 1.4, one has:

$$\left\|\mathbf{G}_\varepsilon\right\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \quad \leq C\left(\left\|\boldsymbol{g}\right\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)}+\left\|\boldsymbol{v}_0\right\|_{\mathbf{H}^1(\Omega)}\right).$$

So that

$$\left|b\left(\mathbf{G}_\varepsilon(t),\boldsymbol{u}_m(t),\,A\boldsymbol{u}_m(t)\right)\right| \quad \leq C\left\|\boldsymbol{u}_m(t)\right\|^2+\frac{\nu}{8}\left|A\boldsymbol{u}_m(t)\right|^2.$$

Finally,

$$\begin{aligned}\left|b\left(\boldsymbol{u}_m(t),\mathbf{G}_\varepsilon(t),\,A\boldsymbol{u}_m(t)\right)\right| &\leq \left\|\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\mathbf{G}_\varepsilon(t)\right\|_{\mathbf{L}^4(\Omega)}\left|A\boldsymbol{u}_m(t)\right|\\&\leq C\left\|\boldsymbol{u}_m(t)\right\|^2\left\|\mathbf{G}_\varepsilon(t)\right\|^2_{\mathbf{H}^2(\Omega)}+\frac{\nu}{8}\left|A\boldsymbol{u}_m(t)\right|^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\left\|\boldsymbol{u}_m(t)\right\|^2+\nu\left|A\boldsymbol{u}_m(t)\right|^2\leq\frac{C}{\nu}\left|\boldsymbol{f}_\varepsilon(t)\right|^2+C\left[\left\|\boldsymbol{u}_m(t)\right\|^4+\left\|\boldsymbol{u}_m(t)\right\|^2\left(1+\left\|\mathbf{G}_\varepsilon(t)\right\|^2_{\mathbf{H}^2(\Omega)}\right)\right].$$

Let

$$\sigma_m(t)=C\left[\left\|\boldsymbol{u}_m(t)\right\|^2+\left(1+\left\|\mathbf{G}_\varepsilon(t)\right\|^2_{\mathbf{H}^2(\Omega)}\right)\right].$$

One knows that

$$\sigma_m(t)\in\mathbf{L}^1(0,T);$$

so that, according to the Gronwall lemma and (24), one has:

$$\boldsymbol{u}_m\quad\in\ \mathbf{L}^\infty(0,T;V)\cap\mathbf{L}^2\left(0,T;\mathbf{H}^2(\Omega)\right),\tag{25}$$

and $\{\boldsymbol{u}_m\}$ is an equibounded sequence in $\mathbf{L}^\infty(0,T;V)\cap\mathbf{L}^2\left(0,T;\mathbf{H}^2(\Omega)\right)$.

2.4 *Estimates III*

Let us multiply (20) by $g'_{jm}(t)$ and sum over j from 1 to m. Then

$$\begin{aligned}\left|\boldsymbol{u}'_m(t)\right|^2=\ &\nu\left(A\boldsymbol{u}_m(t),\boldsymbol{u}'_m(t)\right)-b\left(\boldsymbol{u}_m(t),\boldsymbol{u}_m(t),\boldsymbol{u}'_m(t)\right)\\&-b\left(\mathbf{G}_\varepsilon(t),\boldsymbol{u}_m(t),\boldsymbol{u}'_m(t)\right)-b\left(\boldsymbol{u}_m(t),\mathbf{G}_\varepsilon(t),\boldsymbol{u}'_m(t)\right)+\left(\boldsymbol{f}_\varepsilon,\boldsymbol{u}'_m(t)\right).\end{aligned}$$

From this, one deduces that

$$\begin{aligned}\left|\boldsymbol{u}'_m(t)\right|^2\leq\ &\nu\left|A\boldsymbol{u}_m(t)\right|\left|\boldsymbol{u}'_m(t)\right|+C\left\|\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left|\boldsymbol{u}'_m(t)\right|\\&+C\left\|\mathbf{G}_\varepsilon(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left|\boldsymbol{u}'_m(t)\right|\\&+C\left\|\boldsymbol{u}_m(t)\right\|_{\mathbf{L}^4(\Omega)}\left\|\nabla\mathbf{G}_\varepsilon(t)\right\|_{\mathbf{L}^4(\Omega)}\left|\boldsymbol{u}'_m(t)\right|+\left|\boldsymbol{f}_\varepsilon(t)\right|\left|\boldsymbol{u}'_m(t)\right|\end{aligned}$$

Using the Gagliardo-Nirenberg inequality, estimates (25) and (19), and lemma 1.4 giving the estimate of $\mathbf{G}_\varepsilon$, one deduces that

$$\boldsymbol{u}'_m\quad\in\ \mathbf{L}^2(0,T;\mathbf{H}),\tag{26}$$

and $\{\boldsymbol{u}'_m\}$ is an equibounded sequence in $\mathbf{L}^2(0,T;H)$.

2.5 *Taking the limit.*

It is a consequence of the above estimates that the sequence $\boldsymbol{u}_m$ has a subsequence $\boldsymbol{u}_m$, the same notation being used to avoid unnecessary notation overload:

$$\boldsymbol{u}_m\rightharpoonup\boldsymbol{u}\text{ weakly*}\quad\text{in}\quad\mathbf{L}^\infty(0,T;V),\tag{27}$$

$$\boldsymbol{u}_m\rightharpoonup\boldsymbol{u}\text{ weakly}\quad\text{in}\quad\mathbf{L}^2\left(0,T;\mathbf{H}^2(\Omega)\right),\tag{28}$$

$$\boldsymbol{u}'_m\rightharpoonup\boldsymbol{u}'\text{ weakly}\quad\text{in}\quad\mathbf{L}^2(0,T;\mathbf{H}).\tag{29}$$

But we have a compact embedding

$$\left\{ \boldsymbol{v} \in \mathbf{L}^2\left(0,T;\mathbf{H}^2\left(\Omega\right)\cap V\right),\ V' \in \mathbf{L}^2\left(0,T;\mathbf{H}\right)\right\} \underset{compact}{\hookrightarrow} \mathbf{L}^2\left(0,T;V\right)$$

So that

$$\boldsymbol{u}_m \to \boldsymbol{u} \text{ strongly} \quad \text{in} \quad \mathbf{L}^2\left(0,T;V\right) \text{ and a.e. in } Q_T \tag{30}$$

Let m_0 be fixed and $\boldsymbol{v} \in \langle \boldsymbol{w}_1,\boldsymbol{w}_2,...,\boldsymbol{w}_{m_0}\rangle$. Let m tend towards $+\infty$ in (20). Then

$$\begin{aligned}\left(\boldsymbol{u}'\left(t\right),\boldsymbol{v}\right)+\nu\left(\left(\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)\right) \quad &+b\left(\boldsymbol{u}\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)+b\left(\boldsymbol{u}\left(t\right),\mathbf{G}_\varepsilon\left(t\right),\boldsymbol{v}\right)\\ &+b\left(\mathbf{G}_\varepsilon\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)=\left(\boldsymbol{f}_\varepsilon\ \left(t\right),\boldsymbol{v}\right),\end{aligned}$$

This last relation being valid for all m_0, it remains true for all $\boldsymbol{v} \in \langle \boldsymbol{w}_1,\boldsymbol{w}_2,...,\boldsymbol{w}_m\rangle$, $\forall m \in \mathbb{N}^*$.

Finally let $\boldsymbol{v} \in V$. There exists $\boldsymbol{v}_m \in \langle \boldsymbol{w}_1,\boldsymbol{w}_2,...,\boldsymbol{w}_m\rangle$ such that $\boldsymbol{v}_m \to \boldsymbol{v}$ in V and

$$\begin{aligned}&\left(\boldsymbol{u}'\left(t\right),\boldsymbol{v}\right)+\nu\left(\left(\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)\right)+b\left(\boldsymbol{u}\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)\\ &+b\left(\boldsymbol{u}\left(t\right),\mathbf{G}_\varepsilon\left(t\right),\boldsymbol{v}\right)+b\left(\mathbf{G}_\varepsilon\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{v}\right)=\left(\boldsymbol{f}_\varepsilon\ \left(t\right),\boldsymbol{v}\right)\end{aligned} \tag{31}$$

Now let us note that for all $t \in [0,T]$,

$$\boldsymbol{u}_m\left(t\right) \to \boldsymbol{u}\left(t\right) \quad \text{weakly in} \quad V,$$

and thus

$$\boldsymbol{u}_m\left(0\right) = \boldsymbol{u}_{0m} \to \boldsymbol{u}\left(0\right) \quad \text{weakly in} \quad V.$$

Since

$$\boldsymbol{u}_{0m} \to \boldsymbol{u}_0^\varepsilon \qquad \text{in} \quad V,$$

we have:

$$\boldsymbol{u}\left(0\right) = \boldsymbol{u}_0^\varepsilon.$$

2.6 *Existence of pressure.*

From (31), one has, for all $\boldsymbol{v} \in V$,

$$\left\langle \boldsymbol{u}' - \nu\triangle\boldsymbol{u} + B\left(\boldsymbol{u},\boldsymbol{u}\right) + B\left(\boldsymbol{u},\mathbf{G}_\varepsilon\right) + B\left(\mathbf{G}_\varepsilon,\boldsymbol{u}\right) - \boldsymbol{f}_\varepsilon\ ,\ \boldsymbol{v}\right\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = 0.$$

Consequently, there exists a unique function p of $L^2\left(0,T\right)$ satisfying (16) and such that :

$$p \in L^2\left(0,T;\mathbf{H}^1\left(\Omega\right)\right).$$

This ends the proof of theorem 2.1.□

3 Uniqueness Theorem

Theorem 3.1 *Problem (16) has a unique solution.*

Proof.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two solutions satisfying the hypotheses of theorem 2.1 and let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. Then one has

$$\frac{\partial \boldsymbol{w}}{\partial t} - \nu \triangle \boldsymbol{w} + \boldsymbol{w}.\nabla \boldsymbol{u} + \boldsymbol{v}.\nabla \boldsymbol{w} + \boldsymbol{w}.\nabla \mathbf{G}_\varepsilon + \mathbf{G}_\varepsilon.\nabla \boldsymbol{w} = \mathbf{0}$$

Multiplying by $\boldsymbol{w}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\boldsymbol{w}(t)\right|^2 + \nu\left\|\boldsymbol{w}(t)\right\|^2 = \quad & -\left(\boldsymbol{w}.\nabla\boldsymbol{u},\boldsymbol{w}\right) - \left(\boldsymbol{v}.\nabla\boldsymbol{w},\boldsymbol{w}\right) \\ & -\left(\boldsymbol{w}.\nabla\mathbf{G}_\varepsilon,\boldsymbol{w}\right) - \left(\mathbf{G}_\varepsilon.\nabla\boldsymbol{w},\boldsymbol{w}\right) \end{aligned}$$

But $b\left(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w}\right) = 0$ and $b\left(\mathbf{G}_\varepsilon,\boldsymbol{w},\boldsymbol{w}\right) = 0$. This yields

$$\frac{1}{2}\frac{d}{dt}\left|\boldsymbol{w}(t)\right|^2 + \nu\left\|\boldsymbol{w}(t)\right\|^2 = \quad -b\left(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w}\right) - b\left(\boldsymbol{w},\mathbf{G}_\varepsilon,\boldsymbol{w}\right).$$

One then integrates with respect to t and we get

$$\frac{1}{2}\left|\boldsymbol{w}(t)\right|^2 + \nu\int_0^t \left\|\boldsymbol{w}(s)\right\|^2 ds = -\int_0^t b\left(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w}\right)\, ds - \int_0^t b\left(\boldsymbol{w},\mathbf{G}_\varepsilon,\boldsymbol{w}\right)\, ds.$$

Since

$$\begin{aligned} \left|\int_0^t b\left(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w}\right) ds\right| \quad & \leq \quad C_1\int_0^t \left\|\boldsymbol{w}(s)\right\|_{\mathbf{L}^4(\Omega)} \left\|\boldsymbol{u}(s)\right\|_{\mathbf{L}^2(\Omega)} ds \\ & \leq \quad C_2\int_0^t \left|\boldsymbol{w}(s)\right| \;\; \left\|\boldsymbol{w}(s)\right\| \left\|\boldsymbol{u}(s)\right\| ds \\ & \leq \quad \frac{\nu}{2}\int_0^t \left\|\boldsymbol{w}(s)\right\|^2 ds + C_3\int_0^t \left|\boldsymbol{w}(s)\right|^2 \left\|\boldsymbol{u}(s)\right\|^2 ds. \end{aligned}$$

and, by the same way,

$$\int_0^t b\left(\boldsymbol{w},\mathbf{G}_\varepsilon,\boldsymbol{w}\right) ds \quad \leq \quad \frac{\nu}{2}\int_0^t \left\|\boldsymbol{w}(s)\right\|^2 ds + C_4\int_0^t \left|\boldsymbol{w}(s)\right|^2 \left|\nabla\mathbf{G}_\varepsilon(s)\right|^2 ds.$$

it follows that

$$\left|\boldsymbol{w}(t)\right|^2 \leq C_5\int_0^t \left|\boldsymbol{w}(s)\right|^2 \left(\left|\nabla\mathbf{G}_\varepsilon(s)\right|^2 + \left\|\boldsymbol{u}(s)\right\|^2\right) ds$$

Thanks to the Gronwall lemma, one deduces $\boldsymbol{w} = 0$.$\square$

4 Existence of strong reproductive solution

We first recall results obtained by Kaniel et Shinbrot [5] in the study of the following problem :

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\triangle\boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad Q_T \\ \text{div }\boldsymbol{u} = 0 & \text{in} \quad Q_T \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma_T \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega \end{cases} \tag{32}$$

where Ω is an open and bounded domain of $\mathbb{R}^3$, with a smooth boundary Γ.

The following result establishes the property of a reproductive solution

Theorem 4.1. *Let $T>0$, and $\boldsymbol{f} \in \mathcal{B}_{R,T}$ with $\boldsymbol{f}$ small enough. Then, there exists an unique function $\boldsymbol{u}_0$, independent of t, with $\nabla \boldsymbol{u}_0 \in \mathcal{B}_{R,T}$ and such that the solution of (32) reproduces its initial value at $t = T$:*

$$\boldsymbol{u}(\boldsymbol{x}, T) = \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}),$$

where

$$\mathcal{B}_{R,T} = \left\{ \boldsymbol{u} \in \mathbf{L}^\infty\left(0, T; \mathbf{L}^2(\Omega)\right) \ : \ \|\boldsymbol{u}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq R \right\}.$$

We begin by recalling the following lemma.

Lemma 4.2. *If*

$$\boldsymbol{u} \in \mathbf{L}^2\left(0, T; \mathbf{H}^2(\Omega) \cap V\right) \text{ and } \boldsymbol{u}' \in \mathbf{L}^2(0, T; \mathbf{H})$$

then

$$\boldsymbol{u} \in C([0, T]; V)$$

and

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 = -2\left(\boldsymbol{u}'(t), \triangle \boldsymbol{u}(t)\right). \square$$

Now, let

$$\boldsymbol{v}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}, \quad \boldsymbol{w}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}, \quad \boldsymbol{g} \in \mathbf{H}^{3/2,3/4}(\Sigma_T) \tag{33}$$

with

$$\boldsymbol{g}.\boldsymbol{n} = 0 \text{ on } \quad \Sigma_T \quad \text{ and } \quad \boldsymbol{v}_0(\boldsymbol{x}) = \boldsymbol{w}_0(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}, 0) \quad \boldsymbol{x} \in \Gamma. \tag{34}$$

With these assumptions, it follows from theorem 2.1 that system (1), with data $(\boldsymbol{v}_0, \boldsymbol{g})$, (respectively $(\boldsymbol{w}_0, \boldsymbol{g})$), has an unique solution

$$\boldsymbol{v} \in \mathbf{L}^2\left(0, T; \mathbf{H}^2(\Omega) \cap \mathbf{H}\right) \cap \mathbf{L}^\infty\left(0, T; \mathbf{H}^1(\Omega)\right) \text{ and } \boldsymbol{v}' \in \mathbf{L}^2(0, T; \mathbf{H}),$$

(respectively

$$\boldsymbol{w} \in \mathbf{L}^2\left(0, T; \mathbf{H}^2(\Omega) \cap \mathbf{H}\right) \cap \mathbf{L}^\infty\left(0, T; \mathbf{H}^1(\Omega)\right) \text{ and } \boldsymbol{w}' \in \mathbf{L}^2(0, T; \mathbf{H})\text{).}$$

Let us now set $\mathbf{z} = \boldsymbol{v} - \boldsymbol{w}$. Then

$$\begin{cases} \dfrac{\partial \mathbf{z}}{\partial t} - \nu \triangle \mathbf{z} + \boldsymbol{w}.\nabla \mathbf{z} + \mathbf{z}.\nabla \boldsymbol{v} + \nabla r = \mathbf{0} & \text{in} \quad Q_T, \\ \text{div } \mathbf{z} = 0 & \text{in} \quad Q_T, \\ \mathbf{z} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{z}(0) = \boldsymbol{v}_0 - \boldsymbol{w}_0 & \text{in} \quad \Omega. \end{cases} \tag{35}$$

where $r = p - q$ (q being the pressure corresponding to $\boldsymbol{w}$).

Lemma 4.3. *If*

$$\max \left(\|\boldsymbol{v}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))}, \|\boldsymbol{w}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \right) \leq M \tag{36}$$

under the assumptions (33) and (34) with $0 < M << 1$, *then*

$$\frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu \|\mathbf{z}(t)\|^2 \leq 0 \tag{37}$$

and thus, for all $t \in [0, T]$,

$$\|\boldsymbol{v}(t) - \boldsymbol{w}(t)\| \leq \|\boldsymbol{v}_0 - \boldsymbol{w}_0\| \exp(-\nu t). \tag{38}$$

Proof.

Let P: $\mathbf{L}^2(\Omega) \to \mathbf{H}$, be the orthogonal projection operator. Then

$$\forall \varphi \in \mathbf{H}, (\nabla r, \varphi) = 0.$$

In particular, let us multiply (35) by $P\triangle \mathbf{z} = A\mathbf{z}$:

$$\frac{1}{2}\frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu |A\mathbf{z}|^2 = -(\boldsymbol{w}.\nabla \mathbf{z}, A\mathbf{z}) - (\mathbf{z}.\nabla \boldsymbol{v}, A\mathbf{z})$$

But

$$\begin{aligned} |(\boldsymbol{w}.\nabla \mathbf{z}, A\mathbf{z})| &\leq \|\boldsymbol{w}\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^4(\Omega)} |A\mathbf{z}| \\ &\leq C \|\boldsymbol{w}\| |A\mathbf{z}|^2 \end{aligned}$$

and

$$\begin{aligned} |(\mathbf{z}.\nabla \boldsymbol{v}, A\mathbf{z})| &\leq \|\mathbf{z}\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{v}\| |A\mathbf{z}| \\ &\leq C \|\boldsymbol{v}\| |A\mathbf{z}|^2 . \end{aligned}$$

So that if

$$C \left(\|\boldsymbol{v}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\boldsymbol{w}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \right) \leq \frac{\nu}{2}$$

then

$$\frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu \|\mathbf{z}(t)\|^2 \leq 0$$

and one deduces (38). $\square$

4.1 The main result

Lemma 4.4. *Suppose that* $\mathbf{g}$ *and* $\mathbf{v}_0$ *satisfy hypotheses (4)-(5) and (9). Let us suppose moreover that* $\mathbf{f}_\varepsilon \in \mathbf{L}^\infty\left(0,T;\mathbf{L}^2(\Omega)\right)$ *and that*

$$\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} \leq \alpha \tag{39}$$

$$\|\boldsymbol{f}_\varepsilon\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq K \tag{40}$$

with $\alpha > 0$ and $0 < K << 1$. *Then, if* $\boldsymbol{u}$ *is the solution given by theorem 2.1, one has:*

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq M \tag{41}$$

Remark 4.5. Let us recall that

$$\boldsymbol{u}_0 = \boldsymbol{v}_0 - \mathbf{G}_\varepsilon(.,0)$$

Consequently, if hypothesis (39) takes place, one has from lemma 1.4 :

$$\begin{aligned}
\|\boldsymbol{u}_0\| &\leq \|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{G}_\varepsilon(.,0)\|_{\mathbf{H}^1(\Omega)} \\
&\leq \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} + L\left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)}\right) \\
&\leq \alpha(L+1) = M.\Box
\end{aligned}$$

Proof of lemma 4.4. (see Batchi [5])

Let us multiply (16) by A$\boldsymbol{u}$ and integrate on Ω :

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 + \nu|A\boldsymbol{u}|^2 \leq \quad & \int_\Omega \boldsymbol{f}_\varepsilon\ .A\boldsymbol{u}dx - \int_\Omega(\boldsymbol{u}.\nabla\boldsymbol{u}).A\boldsymbol{u}dx \\
& - \int_\Omega(\boldsymbol{u}.\nabla\mathbf{G}_\varepsilon).A\boldsymbol{u}dx - \int_\Omega(\mathbf{G}_\varepsilon.\nabla\boldsymbol{u}).A\boldsymbol{u}dx
\end{aligned}$$

But

$$\begin{aligned}
\left|\int_\Omega(\boldsymbol{u}.\nabla\boldsymbol{u}).A\boldsymbol{u}\ dx\right| &\leq \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{u}\|\,|A\boldsymbol{u}| \\
&\leq C_1\|\boldsymbol{u}\|\,|A\boldsymbol{u}|^2,
\end{aligned}$$

where C_1 is such that $\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \leq C_1|A\boldsymbol{u}|$.

In the same way, one also has

$$\left|\int_\Omega(\boldsymbol{u}.\nabla\mathbf{G}_\varepsilon).A\boldsymbol{u}dx\right| \leq C_1\|\nabla\mathbf{G}_\varepsilon\|_{\mathbf{L}^2(\Omega)}\ |A\boldsymbol{u}|^2$$

But thanks to the lemma 1.4, one knows that

$$\mathbf{G}_\varepsilon \in \mathbf{L}^\infty\left(0,T;\mathbf{H}^1(\Omega)\right)$$

and

$$
\begin{aligned}
\|\nabla \mathbf{G}_\varepsilon\|_{\mathbf{L}^2(\Omega)} &\leq C_2 \|\mathbf{G}_\varepsilon\|_{\mathbf{H}^{2,1}(Q_T)} \\
&\leq C_2 L \left(\|\boldsymbol{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\boldsymbol{v}_0\|_{\mathbf{H}^1(\Omega)} \right) \\
&\leq C_3 \alpha.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\left| \int_\Omega (\mathbf{G}_\varepsilon . \nabla \boldsymbol{u}) . A\boldsymbol{u} dx \right| &\leq \|\mathbf{G}_\varepsilon\|_{\mathbf{L}^4(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^4(\Omega)} |A\boldsymbol{u}| \\
&\leq C_4 \|\mathbf{G}_\varepsilon\|_{\mathbf{H}^1(\Omega)} |A\boldsymbol{u}| \|\nabla \boldsymbol{u}\|^{1/2}_{\mathbf{L}^2(\Omega)} \left\|\nabla^2 \boldsymbol{u}\right\|^{1/2}_{\mathbf{L}^2(\Omega)} \\
&\leq C_5 \alpha \|\boldsymbol{u}\|^{1/2} |A\boldsymbol{u}|^{3/2} \\
&\leq C_5 \alpha \sqrt{C_6} |A\boldsymbol{u}|^2 ,
\end{aligned}
$$

with $\|\boldsymbol{u}\| \leq C_6 |A\boldsymbol{u}|$.

Thus,

$$
\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}\|^2 + \nu |A\boldsymbol{u}|^2 \leq |\boldsymbol{f}_\varepsilon| \; |A\boldsymbol{u}| + C_1 \|\boldsymbol{u}\| \; |A\boldsymbol{u}|^2 + C_1 C_3 \alpha |A\boldsymbol{u}|^2 + C_5 \alpha \sqrt{C_6} |A\boldsymbol{u}|^2 . \tag{42}
$$

Let $\varphi(t) = \|\boldsymbol{u}(t)\|$

i) Let us first suppose that $\|\boldsymbol{u}_0\| < M$.

Let $t_0 > 0$ be the smallest $t > 0$ such that $\varphi(t_0) = M$. According to (41), one then has

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} + \nu |A\boldsymbol{u}(t_0)|^2 \leq \quad & K |A\boldsymbol{u}(t_0)| + C_1 M |A\boldsymbol{u}(t_0)|^2 \\
& + C_1 C_3 \alpha |A\boldsymbol{u}(t_0)|^2 + C_5 \alpha \sqrt{C_6} |A\boldsymbol{u}(t_0)|^2 .
\end{aligned}
$$

Let us choose α sufficiently small and K such that

$$
K = \frac{\nu}{8} \frac{1}{C_6} M, \qquad \left(C_1 M + C_1 C_3 \alpha + C_5 \alpha \sqrt{C_6} \right) \leq \frac{3\nu}{8}
$$

Then

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} + \nu |A\boldsymbol{u}(t_0)|^2 &\leq \frac{\nu}{8} \frac{1}{C_6} M |A\boldsymbol{u}(t_0)| + \frac{3\nu}{8} |A\boldsymbol{u}(t_0)|^2 \\
\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} + \nu |A\boldsymbol{u}(t_0)|^2 &\leq \frac{\nu}{8} \frac{1}{C_6} \|\boldsymbol{u}(t_0)\| |A\boldsymbol{u}(t_0)| + \frac{3\nu}{8} |A\boldsymbol{u}(t_0)|^2 \\
\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} + \nu |A\boldsymbol{u}(t_0)|^2 &\leq \frac{\nu}{2} |A\boldsymbol{u}(t_0)|^2 .
\end{aligned}
$$

Thus

$$
\frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} + \nu |A\boldsymbol{u}(t_0)|^2 \leq 0
$$

which implies that

$$
\frac{d}{dt} \|\boldsymbol{u}(t)\|^2_{t=t_0} \leq 0
$$

Consequently, there exists $t^* \in [0, t_0[$ such that

$\varphi(t^*) > \varphi(t_0)$, in contradiction with the definition of t_0.

Therefore

$$\forall t \in [0, T], \ \ \varphi(t) < M.$$

ii) Suppose now that $\|\boldsymbol{u}_0\| = M$.

According to the above calculations, one verifies that $\varphi'(0) < 0$ and thus there exists $t^* > 0$ such that

$$\forall t \in \left]0, t^*\right], \varphi(t) < M.$$

Repeating the reasoning made in i), one shows that on $[t^*, T]$, $\varphi(t) < M$, and this ends the proof.□

Remark 4.6. From now on, we assume that $\boldsymbol{g}$ does not dependent on time. More precisely, it is supposed that

$$\boldsymbol{g} \in \mathbf{H}^{3/2}(\Gamma), \quad \boldsymbol{g}.\boldsymbol{n} = 0 \text{ on } \Gamma. \tag{43}$$

One recalls that $\boldsymbol{v}_0 \in \mathbf{H}^1(\Omega)$ satisfies

$$\text{div } \boldsymbol{v}_0 = 0 \text{ in } \Omega, \quad \boldsymbol{v}_0.\boldsymbol{n} = 0 \text{ on } \Gamma \tag{44}$$

and that

$$\boldsymbol{v}_0 = \boldsymbol{g} \quad \text{on } \Gamma. \tag{45}$$

One knows that there exists $\mathbf{G} \in \mathbf{H}^2(\Omega)$ such that

$$\begin{cases} \text{div } \mathbf{G} = 0 & \text{in } \Omega, \\ \mathbf{G} = \boldsymbol{g} & \text{on } \Gamma, \end{cases} \tag{46}$$

with

$$\|\mathbf{G}\|_{\mathbf{H}^2(\Omega)} \leq C \|\boldsymbol{g}\|_{\mathbf{H}^{3/2}(\Gamma)}. \tag{47}$$

Processing as in lemma 1.4, one shows the existence, for all $\varepsilon > 0$, of $\mathbf{G}_\varepsilon \in \mathbf{H}^2(\Omega)$ satisfying (44)-(47) and the estimates:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \ |b(\boldsymbol{v}, \mathbf{G}_\varepsilon, \boldsymbol{v})| \leq \varepsilon \|\boldsymbol{g}\|^2 \tag{48}$$

The right side $\boldsymbol{f}_\varepsilon$ in system (16) then becomes independent of time and satisfies

$$\boldsymbol{f}_\varepsilon \in L^\infty\left(0, T; L^2(\Omega)^2\right) \tag{49}$$

In the same way, $\boldsymbol{u}_0^\varepsilon$ becomes

$$\boldsymbol{u}_0^\varepsilon = \boldsymbol{v}_0 - \mathbf{G}_\varepsilon \tag{50}$$

with $\mathbf{G}_\varepsilon$ depends only on $\boldsymbol{g}$.□

4.2 Reproductive solution result

With these assumptions on $\boldsymbol{g}$ and $\boldsymbol{v}_0$, lemma 4.2 remains naturally valid and one is able to establish the theorem which follows :

Theorem 4.7. *Let* $\boldsymbol{g} \in \mathbf{H}^{3/2}(\Gamma)$ *such that* $\boldsymbol{g}.\boldsymbol{n} = 0$ *on* Γ *and*

$$\|\boldsymbol{g}\|_{\mathbf{H}^{3/2}(\Gamma)} \leq \alpha \tag{51}$$

with $0 < \alpha << 1$ *. Then, there exists* $\boldsymbol{v}_0 \in \mathbf{H}^1(\Omega)$ *such that* $\operatorname{div} \boldsymbol{v}_0 = 0$ *in* Ω *and* $\boldsymbol{v}_0 = \boldsymbol{g}$ *on* Γ*, and such that the solution* $\boldsymbol{v} = \boldsymbol{u} + \mathbf{G}_\varepsilon$ *where* $\boldsymbol{u}$ *is given by theorem 2.1, is reproductive:*

$$\boldsymbol{v}(T) = \boldsymbol{v}(0) = \boldsymbol{v}_0.$$

Proof. Let $\mathbf{G}_\varepsilon \in \mathbf{H}^2(\Omega)$ be the extension of $\boldsymbol{g}$ satisfying(45)-(47) and

$$\boldsymbol{f}_\varepsilon = \nu \triangle \mathbf{G}_\varepsilon - \mathbf{G}_\varepsilon.\nabla \mathbf{G}_\varepsilon$$

Let $\boldsymbol{u}_0^\varepsilon = \boldsymbol{v}_0 - \mathbf{G}_\varepsilon \in V$ and $\boldsymbol{u} \in L^2\left(0, T; \mathbf{H}^2(\Omega)\right) \cap L^\infty(0, T; V)$ be the unique solution of (16). We note that the function $\boldsymbol{v} = \boldsymbol{u} + \mathbf{G}_\varepsilon$ is the unique solution of the initial problem (1). As in the proof of lemma 4.3, it is clear that if $\|\boldsymbol{u}_0^\varepsilon\| < M$, then

$$\sup_{t \in [0,T]} \|\boldsymbol{u}(t)\| \leq M$$

provided that $\|\boldsymbol{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)}$ is sufficiently small, which follows from (49).

Let us define the application

$$\begin{array}{ll} \mathrm{L}: & \boldsymbol{u}_0^\varepsilon \longrightarrow \boldsymbol{u}(.,T) \\ & B_M \longrightarrow B_M \end{array}$$

where $B_M = \{\mathbf{z} \in \boldsymbol{V},\ \|\mathbf{z}\| \leq M\}$;

$\boldsymbol{u}(.,T)$ being the unique solution of (16) at $t = T$.

Moreover, as in remark 4.5, it is clear that if $\|\boldsymbol{v}_0\| \leq \alpha$ and $\|\boldsymbol{w}_0\| \leq \alpha$ then

$$\|\boldsymbol{u}_0^\varepsilon\| \leq M \quad \text{and} \quad \|\boldsymbol{w}_0^\varepsilon\| \leq M,$$

with $\mathbf{y}_0^\varepsilon = \boldsymbol{w}_0 - \mathbf{G}_\varepsilon$.

So that

$$\begin{aligned} \mathrm{L}\boldsymbol{u}_0^\varepsilon(t) - \mathrm{L}\mathbf{y}_0^\varepsilon(t) &= \boldsymbol{u}(t) - \mathbf{y}(t) \\ &= \boldsymbol{u}(t) - \mathbf{G}_\varepsilon - (\mathbf{y}(t) - \mathbf{G}_\varepsilon) \\ &= \boldsymbol{v}(t) - \boldsymbol{w}(t), \end{aligned}$$

and, according to lemma 4.2

$$\begin{aligned}\left\|\mathrm{L}\boldsymbol{u}_0^{\varepsilon}(t)-\mathrm{L}\mathbf{y}_0^{\varepsilon}(t)\right\| &= \|\boldsymbol{v}(T)-\boldsymbol{w}(T)\| \\ &\leq \|\ \boldsymbol{v}_0-\ \boldsymbol{w}_0\|\exp(-\nu T) \\ &\leq \|\boldsymbol{u}_0^{\varepsilon}-\mathbf{y}_0^{\varepsilon}\|\exp(-\nu T)\end{aligned}$$

Thus L is a contraction and has a fixed point.□

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